LINEAR ALGEBRA

23.6.23

Vectors - Matrices:

vector is a quantity having both magnitude & direction

vectors representation

$$V = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

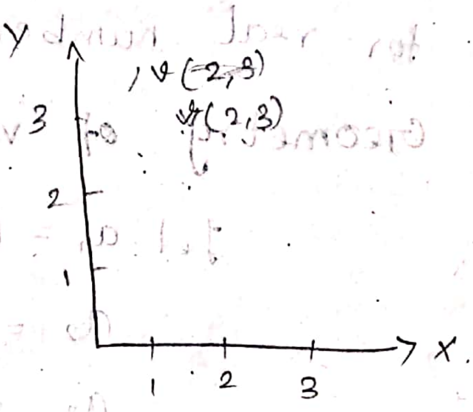
A vector $V = (v_1, v_2 \ldots v_n)$ is a point in R^d where 'd' is a dimension.

R^d - stands for Euclidean space having dimension 'd'.

Example:

$V = (2,3) \in R^2$

$v_1 = 2, \; v_2 = 3$



nxd matrix

Let A be a matrix having 'n' rows and 'd' columns

Vector representation:

$$A = [a_1 ; a_2 ; \dots a_n]$$

Matrix Representation

$$A = \begin{bmatrix} A_{1,1} & A_{1,2} & \dots & A_{1,d} \\ A_{2,1} & A_{2,2} & \dots & A_{2,d} \\ \vdots & \vdots & & \vdots \\ A_{n,1} & A_{n,2} & & A_{n,d} \end{bmatrix}$$

$$a_i = [A_{i,1} \quad A_{i,2} \quad \dots \quad A_{i,d}]$$

and $A_{i,j}$ is the element of the matrix A in the i^{th} row j^{th} column

Here $A \in R^{n \times d}$, where 'R' stands for real numbers.

Geometry of vectors and Matrices:

Let $a_1 = (-0.5, 1.5)$

$a_2 = (1,1)$

$a_3 = (2.5, 0.75)$

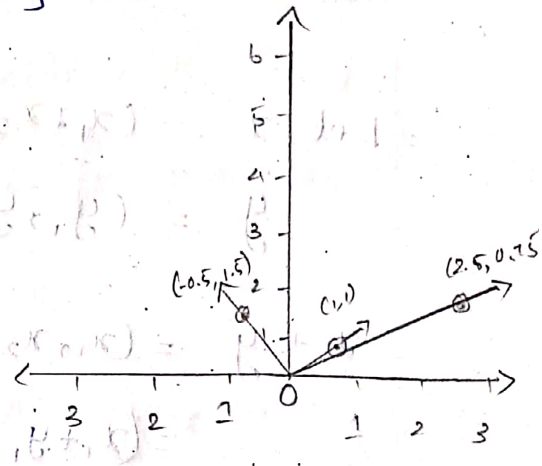
vector representation:

$$A = [a_1 ; a_2 ; a_3]$$

Matrix representation

$$A = \begin{bmatrix} -0.5 & 1.5 \\ 1 & 1 \\ 2.5 & 0.75 \end{bmatrix}$$

Here $n=3$ and $d=2$



$A_{1,1} = -0.5$

$A_{1,2} = 1.5$

$A_{2,1} = 1$

$A_{2,2} = 1$

$A_{3,1} = 2.5$

$A_{3,2} = 0.75$

Transpose operation

$$A^T = [a_1, a_2 \dots a_n] = \begin{bmatrix} A_{1,1}, A_{2,1} \dots A_{n,1} \\ A_{1,2}, A_{2,2} \dots A_{n,2} \\ A_{1,d}, A_{2,d} \dots A_{n,d} \end{bmatrix}$$

Linear equation Representation

Consider the equation

$$3x_1 - 7x_2 + 2x_3 = -2$$

$$-1x_1 + 2x_2 - 5x_3 = 6$$

A is the integrator operator in descrite form

x is the vector of the unknown derivators

y is the vector of meas values of the function to be differentiated

Matrix vector notation is $Ax = b$

Matrix, vector, vector

$$A = \begin{bmatrix} 3 & -7 & 2 \\ -1 & 2 & -5 \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad b = \begin{bmatrix} -2 \\ 6 \end{bmatrix}$$

n = no. of linear equation = 2

d = no. of variables = 3

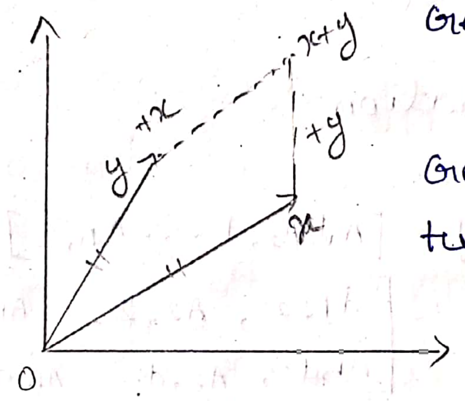
Addition and Multiplication of vectors

Let $x = (x_1, x_2 \dots x_d) \in R^d$

$y = (y_1, y_2 \dots y_d) \in R^d$

$x + y = (x_1, x_2 \dots x_d) + (y_1, y_2 \dots y_d)$

$= (x_1 + y_1 ; x_2 + y_2 ; x_d + y_d) \in R^d$



Geometry of vector addition

Geometric: chaining two vectors together

(i) Vector addition is commutative

(i.e) $x + y = y + x$

consider two matrices $A, B \in R^{n \times d}$

$C = A + B$ is defined by

$C_{ij} = A_{ij} + B_{ij}$ for all i and j

ii) Matrix multiplication

Let $A \in R^{n \times d}$ and $B \in R^{d \times m}$

$C = AB \in R^{n \times m}$ is defined as

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$$C_{i,j} = \sum_{k=1}^{d} A_{i,k} \cdot B_{k,j}$$

where $C_{i,j}$ is the element in i^{th} row and j^{th} column.

Properties of matrix multiplication.

1. Matrix multiplication is association

   (i.e) $(A \cdot B) C = A (BC)$

2. Matrix multiplication is distributive.

   $A(B+C) = AB + AC$

3. Matrix multiplication is ~~not~~ not commutative

   (i.e) $AB \neq BA$

4. Scalar multiplication

   Let 'A' be a given matrix and 'α' be any scalar.

   Then $\alpha A = A \alpha$ and is defined by a new matrix $B = \alpha A$

   Where $B_{ij} = \alpha A_{ij}$

Vector ~~to~~ - vector products

Two type of vector product are available consider column vector $x, y \in \mathbb{R}^d$

$$\therefore x = [x_1, x_2 \dots x_d]$$

$$y = [y_1, y_2 \dots y_d]$$

Inner product (or) Dot product is defined as

$$x^T \cdot y = x \cdot y = [x_1, x_2 \dots x_d] = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$x_1 y_1 + x_2 y_2 + \dots \dots x_d y_d$$

$$\sum_{i=1}^{d} x_i y_i$$

Where x_i is the i^{th} element of x.

Dot product gives signale value and its a linear operator.

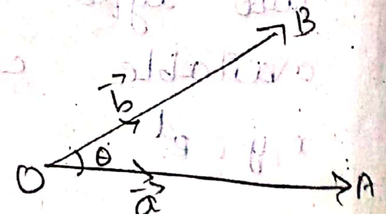
Since it is a linear operator it satifies the following condition.

Let $x, y, z \in R^d$ and α be any scalar

$$\langle \alpha x, y+z \rangle = \alpha \langle x, y+z \rangle$$
$$= \alpha(\langle x, y \rangle + \langle x, z \rangle)$$

Geometry meaning of Dot product

$$\vec{a} \cdot \vec{b} = |a| |b| \cos\theta$$



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Geometry of Dot product

consider two vectors

$U = (3/5, 4/5) \quad V = (2,1)$

Let θ be the angle between U and V

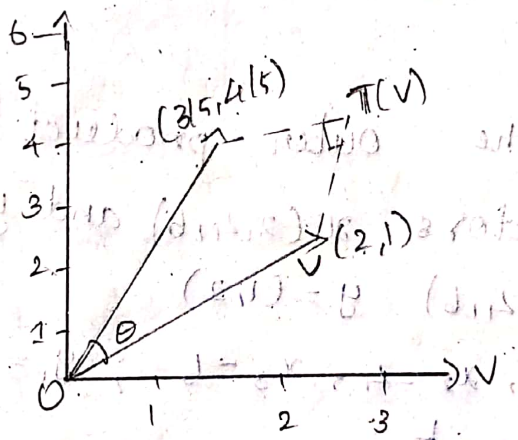
$\langle U, V \rangle$ = Lenght (U) . Lenght (V) $\cos\theta$

$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos\theta$

$$\text{Lenght}(U) = \sqrt{\left(\frac{3}{5}\right)^2 + \left(\frac{4}{5}\right)^2} = \sqrt{\frac{9}{25} + \frac{16}{25}}$$

$$= \sqrt{\frac{25}{25}} = \sqrt{1} = 1$$

$$\text{Lenght}(V) = \sqrt{2^2 + 1^2} = \sqrt{4+1} = \sqrt{5}$$



$\langle U, V \rangle$ represent lenght of V projected out to the line ~~together~~ through U.

$\therefore \langle U, V \rangle$ = lenght $(\pi\, U(V))$

$= \pi\, U(V) \; \| [\text{normal of } \pi_U(V)]$

Outer product

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- Consider two coloumns vectors $x \in R^n$ and $y \in R^d$

Then their outer product is given as

$$y^T x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} [y_1, y_2 \cdots y_d]$$

$$\begin{bmatrix} x_1 y_1 & x_1 y_2 & \cdots & x_1 y_d \\ x_2 y_1 & x_2 y_2 & \cdots & x_2 y_d \\ \vdots & \vdots & & \vdots \\ x_n y_1 & x_n y_2 & & x_n y_d \end{bmatrix}$$

1) Find the outer product for two vectors $x(2,4,6)$ and $y=(1,2)$

$x=(2,4,6)$ $y=(1,2)$

$x_1 = 2$, $x_2 = 4$, $x_3 = 6$; $y_1 = 1$, $y_2 = 2$

Outer product

$$y^T x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} [y_1, y_2] = \begin{bmatrix} x_1 y_1 & x_1 y_2 \\ x_2 y_1 & x_2 y_2 \\ x_3 y_1 & x_3 y_2 \end{bmatrix}$$

$$\begin{bmatrix} 2\times 1 & 2\times 2 \\ 4\times 1 & 4\times 2 \\ 6\times 1 & 6\times 2 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ 4 & 8 \\ 6 & 12 \end{bmatrix}$$

(2) Find the characteristics polynominal and eigen values of the matrix $A = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & -1 \\ 2 & -1 & 0 \end{pmatrix}$

Soln:

Characteristic equation is $|A - \lambda I| = 0$

$$\left| \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & -1 \\ 2 & -1 & 0 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right| = 0$$

$$\left| \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & -1 \\ 2 & -1 & 0 \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} \right| = 0$$

$$\begin{vmatrix} -\lambda & 1 & 2 \\ 1 & -\lambda & -1 \\ 2 & -1 & -\lambda \end{vmatrix} = 0$$

$\cancel{-\lambda(+\lambda^2+1) - 1(-\lambda-2) + 2(1-2\lambda) = 0}$

$\cancel{-\lambda^3+\lambda+2 = 0}$ $\quad$ $\cancel{-\lambda^3-\lambda+2+2-4\lambda = 0}$

$\cancel{-\lambda^3-3\lambda+2 = 0}$ $\quad$ $\cancel{-\lambda^3-6\lambda+}$

$-\lambda(+\lambda^2+1) - 1(-\lambda+2) + 2(-1+2\lambda) = 0$

$-\lambda^3+\lambda+\lambda-2-2+4\lambda = 0$

$-\lambda^3+6\lambda-4 = 0$

2	1	0	-6	4
	0	2	4	-4
	1	2	-2	0

$\lambda^3-6\lambda+4 = 0$

$(\lambda-2)(\lambda^2-2\lambda-2) = 0$

$(\lambda-2) = 0$ $\quad$ $\lambda^2-2\lambda-2 = 0$

$\lambda = 2$ $\quad$ $(\lambda+1)^2-1-2 = 0$

$(\lambda+1)^2 = 3$

$\lambda+1 = \pm\sqrt{3}$

$\lambda = -1 \pm \sqrt{3}$

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Matrix vector product.

Consider a matrix $A \in R^{n \times d}$

and a vector $x \in R^d$

There produce is defined as

$y = Ax \in R^n$

Here A consists of row vectors

$[a_1, a_2 \dots a_n]$

Matrix vector product is

defined as $y = Ax = \begin{bmatrix} \dots a_1 \dots \\ \dots a_2 \dots \\ \dots a_n \dots \end{bmatrix} x = \begin{bmatrix} \langle a_1, x \rangle \\ \langle a_2, x \rangle \\ \vdots \\ \langle a_n, x \rangle \end{bmatrix}$

NORMS:

Norm means - lenght

1) Euclidean norm of a vector

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Definition:

consider a vector $v = (v_1, v_2 \dots v_d) \in R^d$

Euclidean norm of this vector

v is defined as

$$\|v\| = \sqrt{\sum_{i=1}^{d} v_i^2} = \sqrt{v_1 v_1 + v_2 v_2 + v_3 v_3 + \dots v_n v_n}$$

$$= \sqrt{\langle v, v \rangle}$$

and the notation is $\|\cdot\|$

Norm of a unit vector

A vector v with norm $\|v\| = 1$ is said to be a unit vector

Remark: A vector $x \in R^d$ with $\|x\| = 1$ is said to be norms

Lp norm:

Lp norm is defined by

$$\|v\|_p = \left(\sum_{i=1}^{d} |v_i|^p\right)^{1/p}$$

~~Frobenius Norm~~

Lp norm of vector

Lp norm of a vector v for any parameters $p \in [1, \infty]$ is defined as

$$\|v\|_p = \left(\sum_{i=1}^{d} |v_i|^p\right)^{1/p}$$

Remark: When $p=2$, Norm of a vector is defined as

$$\|v\|_2 = \|v\| = \sqrt{\sum_{i=1}^{d} v_i^2} = \sqrt{\langle v, v \rangle}$$

Forbenius Norm:

Definition: frobenius is defined for a matrix $A \in R^{n \times d}$

$$\|A\| = \|A\|_2 = \max \frac{\|Ax\|}{\|x\|} = \max_{y \in R^n} \frac{\|yA\|}{\|y\|}$$

$\|x\| \neq$ $\|y\| \neq 0$

Linear Independance

Consider a set of k vectors

$x_1, x_2 \ldots x_k \in R^d$

and a set of k scalars

$\alpha_1, \alpha_2, \ldots \alpha_k \in R^1$

Write a new vector $x \in R^d$ as

$x = \alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_k x_k$

$x = \sum_{k=1}^{d} \alpha_k x_k$

For a set of vectors

$x = \{x_1, x_2 \ldots x_k\}$

For any vectors 'z' such that there exists of set of scalars

$\{\alpha_1, \alpha_2 \ldots \alpha_k\}$ to z.

If 'z' can be written in the following form.

$z = \alpha_1 x_1 + \alpha_2 x_2 + \ldots \alpha_k x_k$ then we say that

'z' is linear by dependent on x

NOTE 1:

if 'z' cannot be written with any choice of α's then we say that 'z' is linearly independent on X.

NOTE 2:

All vector $z \in \mathbb{R}^d$ which are linearly dependent on x are said to be in its span.

$$span(x) = \{ z \mid z = \sum_{i=1}^{d} \alpha_i x_i, \alpha_i \in \mathbb{R} \}$$

Definition

Linear dependence

A set of vectors. $X = \{x_1, x_2 \dots x_n\}$ is linearly independent if there is no way to write any vector $x_i \in X$ in the set with scalars $\{\alpha_1, \alpha_2, \dots \alpha_{i-1}, \alpha_{i+1} \dots \alpha_n\}$ as the sum

$x_i = \sum_{i=1}^{n} \alpha_j x_j$ of other vectors in the set $j \neq i$

1) Problem :

Check whether the following vectors in a set $x = \{x_1, x_2\}$

$$x_1 = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} \quad x_2 = \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix}$$ are linearly independent or dependent.

Soln:

Take two vectors

$$z_1 = \begin{bmatrix} -3 \\ -5 \\ 2 \end{bmatrix} \text{ and } z_2 = \begin{bmatrix} 3 \\ 7 \\ 1 \end{bmatrix}$$

Case(i): z_1 is linearly dependent

$$z_1 = \alpha_1 x_1 + \alpha_2 x_2$$

$$\begin{bmatrix} -3 \\ -5 \\ 2 \end{bmatrix} = \alpha_1 \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} + \alpha_2 \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ 3\alpha_1 \\ 4\alpha_1 \end{bmatrix} + \begin{bmatrix} 2\alpha_2 \\ 4\alpha_2 \\ \alpha_2 \end{bmatrix}$$

$$\begin{bmatrix} -3 \\ -5 \\ 2 \end{bmatrix} = \begin{bmatrix} \alpha_1 + 2\alpha_2 \\ 3\alpha_1 + 4\alpha_2 \\ 4\alpha_1 + \alpha_2 \end{bmatrix}$$

$$\alpha_1 + 2\alpha_2 = -3 \quad \text{——} (1)$$

$$3\alpha_1 + 4\alpha_2 = -5 \quad \text{——} (2)$$

$$4\alpha_1 + \alpha_2 = 2 \quad \text{——} (3)$$

Solving equation (1) & (2)

(1) × 3 $\quad 3\alpha_1 + 6\alpha_2 = -9$

$$\begin{array}{r} 3\alpha_1 + 4\alpha_2 = -5 \\ (-) \quad\; (-) \quad\quad\; (+) \\ \hline 2\alpha_2 = -4 \end{array}$$

$\alpha_2 = -4/2$

$\boxed{\alpha_2 = -2}$

Substitute $\alpha_2 = -2$ in (1)

$\alpha_1 + 2(-2) = -3$

$\alpha_1 - 4 = -3$

$\alpha_1 = -3+4$

$\alpha_1 = 1$

Substitute $\alpha_1 = 1$, $\alpha_2 = -2$ in (3)

$4\alpha_1 + \alpha_2 = 2$

$4(1) + (-2) = 2$

$4 - 2 = 2$

$2 = 2$

Equation (3) is satisfied

$z_1 = \alpha_1 x_1 + \alpha_2 x_2$

$= 1.x - 2x_2$

$z_1 = x - 2x_2$

z_1 is linearly dependent

case(ii):

Consider z_2 vector

$z_2 = \alpha_1 x_1 + \alpha_2 x_2$

$\Rightarrow \begin{bmatrix} 3 \\ 7 \\ 1 \end{bmatrix} = \alpha_1 \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} + \alpha_2 \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix}$

$$= \begin{bmatrix} \alpha_1 \\ 3\alpha_1 \\ 4\alpha_1 \end{bmatrix} + \begin{bmatrix} 2\alpha_2 \\ 4\alpha_2 \\ 1\alpha_2 \end{bmatrix} = \begin{bmatrix} \alpha_1 + 2\alpha_2 \\ 3\alpha_1 + 4\alpha_2 \\ 4\alpha_1 + \alpha_2 \end{bmatrix}$$

$\alpha_1 + 2\alpha_2 = 3$ — ①

$3\alpha_1 + 4\alpha_2 = 7$ — ②

$4\alpha_1 + \alpha_2 = 1$ — ③

Solving ① and ②

① × 3 ⇒ $3\alpha_1 + 6\alpha_2 = 9$

$3\alpha_1 + 4\alpha_2 = 7$

$2\alpha_2 = 2$

$\alpha_2 = 1$

Sub $\alpha_1 = 1$, α_2 in ③

$4(1) + (1) = 1$

$4 + 1 = 1$

$5 = 1$

Which is not possible

$\therefore Z_2 = \alpha_1 x_1 + \alpha_2 x_2$

$Z_2 = 1x_1 + 1x_2$ is not possible

$\therefore Z_2$ is linearly independent

Orthogonality

Definition: orthogonal vectors

Two vectors $x, y \in \mathbb{R}^d$ are orthogonal if $\langle x, y \rangle = 0$

Example: Show that two vector

$x = (2, -3, 4, -1, 6)$

$y = (4, 5, 3, -7, -2)$ are orthogonal.

Soln:

Find linear product between x and y

$\langle x, y \rangle = 2 \times 4 + (-3)(5) + 4 \times 3 + (-1)(-7) + 6(-2)$

$= 8 - 15 + 12 + 7 - 12$

$= -7 + 19 - 12$

$= 19 - 19$

$= 0$

Since $\langle x, y \rangle = 0$ the given vectors x and y are orthogonal.

10-07-2023

A matrix $U \in R^{n \times n}$ is said to be orthogonal if all of its columns $[u_1, u_2 \ldots u_n]$ are normalized and are all orthogonal with each other.

$\rightarrow \langle u, v \rangle = \|u\| = 1$

1) Normalization
2) orthogonal

$\Big\} =$ orthonormal

$\langle u, v \rangle = 0$

$\langle u_i, u_j \rangle = 1$ if $i = j$ → Normalize

$\langle u_i, u_j \rangle = 0$ if $i \neq j$ → orthogonal

NOTE1: orthogonal matrices are norm processing with multiplication.

$U \in R^{n \times n}$ — orthogonal matrix

$x \in R^n$ — vector

$\|U x\| = \|x\|$

NOTE 2: Let $U \in R^{n \times n}$ be an orthogonal matrix form a basis.

$U = [u_1, u_2 \ldots u_n]$

1) For any vector $x \in R^n$, there exists a set of scalars

$$x = \alpha_1 u_1 + \alpha_2 u_2 + \ldots \alpha_n u_n$$

$$x = \sum^{n} \alpha_i u_i$$

Note:

$$\|x\|^2 = \alpha_1^2 + \alpha_2^2 + \dots + \alpha_n^2 = \sum_{i=1}^{n} \alpha_i^2$$

$$U = (U_1, U_2) \in R^{2\times 2}$$

$$\alpha_1 = 2 \quad \alpha_2 = -3$$

$$x = 2U_1 - 3U_2$$

$$\|x\|^2 = \alpha_2^2 + \alpha_2^2$$

$$= (2)^2 + (-3)^2$$

$$= 4 + 9$$

$$= \cancel{18}\ 13$$

Rank of a matrix:

Definition: A number r is said to be rank of a matrix A. If the following properties are statisfied.

(i) There exists atleast one square submatrix A of order r, whose determinant value $\neq 0$.

ii) If the matrix A contains any square sub matrix of order r+1, then the determinant of that matrix is zero.

Ra

Q1 Find the rank of the matrix

$$A = \begin{bmatrix} 2 & 3 & 4 \\ 0 & 1 & 2 \\ -2 & 0 & 1 \end{bmatrix}$$

AM

Soln:

Order of matrix $A = 3 \times 3$

$r = 3$

$$|A| = \begin{bmatrix} 2 & 3 & 4 \\ 0 & 1 & 2 \\ -2 & 0 & 1 \end{bmatrix}$$

$= 2(1-0) - 3(0+4) + 4(0+2)$

$= 2(1) - 3(4) + 4(2)$

$= 2 - 12 + 8$

$= \cancel{2}\,10 - 12$

$= -2 \neq 0$

$\therefore \rho(A) = 3$

2) Find the rank of the matrix:

$$B = \begin{bmatrix} 3 & 4 & -3 \\ 5 & 5 & -3 \\ 2 & 1 & 0 \end{bmatrix}$$

Soln: order of B is 3×3

$$|B| = \begin{vmatrix} 3 & 4 & -3 \\ 5 & 5 & -3 \\ 2 & 1 & 0 \end{vmatrix} = 3\begin{vmatrix} 5 & -3 \\ 1 & 0 \end{vmatrix} - 4\begin{vmatrix} 5 & -3 \\ 2 & 0 \end{vmatrix} - 3\begin{vmatrix} 5 & 5 \\ 2 & 1 \end{vmatrix}$$

$= 3(0+3) - 4(0+6) - 3(5-10)$

$= 3(3) - 4(6) - 3(-5)$

$= 9 - 24 + 15$

$= 24 - 24$

$|B| = 0$

Since $|B| = 0$, $\rho(3) \neq 3$

Consider 2×2 submatrix

$$B_1 = \begin{pmatrix} 3 & 4 \\ 5 & 5 \end{pmatrix}_{2\times 2}$$

$= 15 - 20$

$= -5$

$\neq 0$

Since $|B_1| \neq 0$, $\rho(B) = 2$

Eigen values and Eigen vectors

(or)

Characteristics ~~and~~ roots and characteristics vect

Step 1: From the characteristics equation

$$(A - \lambda I) X = 0$$

A = Matrix

λ = Eigen value

I - Identity

X - Characteristics vector

1) Find the eigen values and eigen vectors for the matrix $A = \begin{pmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{pmatrix}$

A

Short cut method to find eigen values

$x^3 - x^2$ (sum of the diagnol elements) +

λ (sum of the co-factors of the diagnols)

$\lambda^3 - \lambda^2$(sum of the diagnol elements) + λ(sum of the co-factors of the diagnol elements) - (value of the determinant)

sum of the diagnol elements (S_1) $= 8+7+3 = 18$

sum of the co-factors the diagnol elements (S_2) $= \begin{vmatrix} 7 & -4 \\ -4 & 3 \end{vmatrix} + \begin{vmatrix} 8 & 2 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 8 & -6 \\ -6 & 7 \end{vmatrix}$

$= (21-16) + (24-4) + (56-36)$

$= 5 + 20 + 20 = 45$

$|A| = \begin{vmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{vmatrix}$ λ

$= 8(21-16) + 6(-18+8) + 2(24-14)$

$= 8(5) + 6(-10) + 2(10)$

$= 40 - 60 + 20$

$= 60 - 60 = 0$ $\quad \lambda^3 S_1 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

$\lambda^3 + 18\lambda^2 + 45\lambda - 0 = 0$

$\lambda(\lambda^2 \mp 18\lambda + 45) = 0$ $\quad 45$

$\lambda = 0, \quad \lambda^2 \mp 18\lambda + 45 = 0$

$(\lambda - 3)(\lambda - 15) = 0$

$\lambda = 0, \lambda = 3, \lambda = 15$

To find the characteristics vectors:

form the ch. equation

$$(A - \lambda I)X = 0$$

$$= \left(\begin{pmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right) \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow (1)$$

Case I: Put $\lambda = 0$ in (1)

$$\begin{pmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$8x_1 - 6x_2 + 2x_3 = 0 \rightarrow (2)$$

$$-6x_1 + 7x_2 + -4x_3 = 0 \rightarrow (3)$$

$$2x_1 - 4x_2 + 3x_3 = 0 \rightarrow (4)$$

Solving ② & ③

$$\begin{array}{cccc} x_2 & x_3 & x_1 & x_2 \\ -6 & 2 & 8 & -6 \\ 7 & -4 & -6 & 7 \end{array}$$

$$\frac{x_1}{24-14} = \frac{x_2}{-12+32} = \frac{x_3}{56-36}$$

$$\frac{x_1}{10} = \frac{x_2}{20} = \frac{x_3}{20} = \frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{2}$$

characteristic vector is $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$

case 2: put $\lambda = 3$ in ①

$$\begin{pmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 5 & -6 & 2 \\ -6 & 4 & -4 \\ 2 & -4 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \longrightarrow ①$$

$$5x_1 - 6x_2 + 2x_3 = 0 \longrightarrow ②$$
$$-6x_1 + 4x_2 - 4x_3 = 0 \longrightarrow ③$$
$$2x_1 - 4x_2 + 0x_3 = 0 \longrightarrow ④$$

Solving ② & ③

$$\begin{array}{cccc} x_2 & x_3 & x_1 & x_2 \\ -6 & 2 & 5 & -6 \\ 4 & -4 & -6 & 4 \end{array}$$

$$\frac{x_1}{24-8} = \frac{x_2}{-12+20} = \frac{x_3}{20-36}$$

$$\frac{x_1}{16} = \frac{x_2}{8} = \frac{x_3}{-16} = \frac{x_1}{2} = \frac{x_1}{1} = \frac{x_3}{-2}$$

characteristic equation $\lambda = \begin{pmatrix} 2 \\ 4 \\ -2 \end{pmatrix}$

case 3: put $\lambda = 15$ in ①

$$\begin{pmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow ①$$

$$\begin{pmatrix} -7 & -6 & 2 \\ -6 & -8 & -4 \\ 2 & -4 & -12 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow ②$$

$-7x_1 - 6x_2 + 2x_3 \rightarrow ②$

$-6x_1 - 8x_2 - 4x_3 \rightarrow ③$

$2x_1 - 4x_2 - 12x_3 \rightarrow ④$

solving ② & ③

x_2	x_3	x_1	x_2
-6	2	-7	-6
-8	-4	-6	-8

$$\frac{x_1}{24+16} = \frac{x_2}{-12-28} = \frac{x_3}{52-36}$$

$$\frac{x_1}{40} = \frac{x_2}{-40} = \frac{x_3}{20} = \frac{x_1}{1} = \frac{x_2}{-1} = \frac{x_3}{1/2}$$

characteristics equation is $X = \begin{pmatrix} 1 \\ -1 \\ 1/2 \end{pmatrix}$

Find the rank of the matrix

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ -1 & 0 & 1 & 2 \\ 2 & -2 & 0 & 1 \\ 3 & 2 & 1 & 3 \end{pmatrix}$$

10M

Solution

Order of A is 4×4

$\therefore$ Rank of A, $\rho(A) \leq 4$.

$$A = \begin{array}{c} \begin{pmatrix} 1 & 2 & 3 & 4 \\ -1 & 0 & 1 & 2 \\ 2 & -2 & 0 & 1 \\ 3 & 2 & 1 & 3 \end{pmatrix} \end{array} \begin{array}{l} R_1 \\ R_2 \\ R_3 \\ R_4 \end{array}$$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 4 & 6 \\ 0 & -6 & -6 & -7 \end{pmatrix} \begin{array}{l} R_1' = R_1 \\ R_2' = R_2 + R_1 \\ R_3' = R_3 - 2R_1 \\ R_4' = R_4 - 3R_1 \end{array}$$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 4 & 6 \\ 0 & 0 & 6 & 11 \\ 0 & 0 & 0 & 3 \end{pmatrix} \begin{array}{l} R_1'' = R_1' \\ R_2'' = R_2' \\ R_3'' = R_3 - 3R_2' \\ R_4'' = R_4' + 2R_2' \end{array}$$

$R_4 : 3 \quad 2 \quad 1 \quad 3$

$3R_1 : 3 \quad 6 \quad 9 \quad 12$

$R_4 - 3R_1 = 0 \quad -4 \quad -8 \quad -9$

$R_3' : 0 \quad -6 \quad -6 \quad 7$

$3R_2' : 0 \quad 6 \quad 12 \quad 18$

$R_3 + 3R_2' : 0 \quad 0 \quad 6 \quad 11$

$0 \quad 0 \quad 0 \quad 3$

This is the required Echelon form

Now, No of non zero rows in the above echelon form is 4

$\therefore$ Rank of matrix $\boxed{\rho(A) = 4.}$

Find the rank of the matrix

10M

$$A = \begin{pmatrix} 1 & -1 & 0 & 2 & 1 \\ 3 & 1 & 1 & -1 & 2 \\ 4 & 0 & 1 & 0 & 3 \\ 9 & -1 & 2 & 3 & 7 \end{pmatrix}$$

Solution:

Order of matrix A is 4×5

$\therefore \rho(A) \leq 4$

$$A = \begin{array}{c} \begin{pmatrix} 1 & -1 & 0 & 2 & 1 \\ 3 & 1 & 1 & -1 & 2 \\ 4 & 0 & 1 & 0 & 3 \\ 9 & -1 & 2 & 3 & 7 \end{pmatrix} \end{array} \begin{array}{l} R_1 \\ R_2 \\ R_3 \\ R_4 \end{array}$$

Put: $R_1' = R_1$; $R_2' = R_2 - 3R_1$

$R_3' = R_3 - 4R_1$, $R_4' = R_4 - 9R_1$

$$A = \begin{pmatrix} 1 & -1 & 0 & 2 & 1 \\ 0 & 4 & 1 & -7 & -1 \\ 0 & 4 & 1 & -8 & -1 \\ 0 & 8 & 2 & -15 & -2 \end{pmatrix} \begin{array}{l} R_1' = R_1 \\ R_2' = R_2 - 3R_1 \\ R_3' = R_3 - 4R_1 \\ R_4 = R_4 - 9R_1 \end{array}$$

$$A = \begin{pmatrix} 1 & -1 & 0 & 2 & 1 \\ 0 & 4 & 1 & -7 & -1 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix} \begin{array}{l} R_1'' = R_1' \\ R_2'' = R_2' \\ R_3'' = R_3' - R_2'' \\ R_4'' = R_4' - 2R_2' \end{array}$$

$$= \begin{pmatrix} 1 & -1 & 0 & 2 & 1 \\ 0 & 4 & 1 & -7 & 1 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{matrix} R_1''' = R_1'' \\ R_2''' = R_2'' \\ R_3''' = R_3'' \\ R_4''' = R_4'' - R_3 \end{matrix}$$

No. of non zeros = 3

$\boxed{\therefore \rho(A) = 3}$